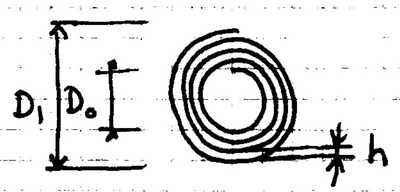
CALCUL EXACTE DE LA LONGUEUR D'UNE BOBINE



Spirale en coordonnées polaires:

$$\rho = \rho(\varphi) = \frac{h}{2\pi}\varphi \qquad \frac{d\rho}{d\varphi} = \frac{h}{2\pi}$$

$$\varphi_0 = \frac{\pi D_0}{h} \quad ; \quad \varphi_1 = \frac{\pi D_1}{h}$$

Longueur d'une courbe en coordonnées polaires:

$$L(\varphi_0, \varphi_1) = \int_{\varphi_0}^{\varphi_1} \sqrt{\rho^2 + \left(\frac{d\rho}{d\varphi}\right)^2}\, d\varphi$$

Donc

$$L(\varphi_0, \varphi_1) = \int_{\varphi_0}^{\varphi_1} \sqrt{\left(\frac{h}{2\pi}\right)^2 \varphi^2 + \left(\frac{h}{2\pi}\right)^2}\, d\varphi = \frac{h}{2\pi}\int_{\varphi_0}^{\varphi_1} \sqrt{\varphi^2+1}\, d\varphi =$$

$$\left[\begin{array}{l} \operatorname{ch}^2 t - \operatorname{sh}^2 t = 1 \iff \operatorname{ch} t = \sqrt{1+\operatorname{sh}^2 t} \\ \text{Changement de variable} \quad x = \operatorname{sh} t \quad ; \quad dx = \operatorname{ch} t\, dt \end{array}\right]$$

$$= \frac{h}{2\pi}\int_{\operatorname{ash}\varphi_0}^{\operatorname{ash}\varphi_1} \sqrt{\operatorname{sh}^2 t + 1}\, \operatorname{ch} t\, dt = \frac{h}{2\pi}\int_{\operatorname{ash}\varphi_0}^{\operatorname{ash}\varphi_1} \operatorname{ch}^2 t\, dt = \frac{h}{2\pi}\int_{\operatorname{ash}\varphi_0}^{\operatorname{ash}\varphi_1} \frac{1}{2}(1+\operatorname{ch}(2t))\, dt =$$

$$\left[\operatorname{ch}^2 t = \frac{1}{2}(1+\operatorname{ch}(2t))\right]$$

$$\left[\operatorname{sh}(2t) = 2\operatorname{sh}(t)\operatorname{ch}(t)\right]$$

$$= \frac{h}{2\pi}\cdot\frac{1}{2}\left(\int_{\operatorname{ash}\varphi_0}^{\operatorname{ash}\varphi_1} dt + \int_{\operatorname{ash}\varphi_0}^{\operatorname{ash}\varphi_1} \operatorname{ch}(2t)\, dt\right) = \frac{h}{2\pi}\cdot\frac{1}{2}\left(t + \frac{1}{2}\operatorname{sh}(2t)\right)\Bigg|_{\operatorname{ash}\varphi_0}^{\operatorname{ash}\varphi_1} =$$

$$\left[\operatorname{ch}(\operatorname{ash}\varphi) = \sqrt{1+\varphi^2}\right]$$

$$= \frac{h}{2\pi}\cdot\frac{1}{2}\left(t + \operatorname{sh}(t)\operatorname{ch}(t)\right)\Bigg|_{\operatorname{ash}\varphi_0}^{\operatorname{ash}\varphi_1} \overset{[t = \operatorname{ash}\varphi]}{=} \frac{h}{2\pi}\frac{1}{2}\left(\operatorname{ash}\varphi + \operatorname{sh}(\operatorname{ash}\varphi)\operatorname{ch}(\operatorname{ash}\varphi)\right)\Bigg|_{\varphi_0}^{\varphi_1} =$$

$$= \frac{h}{2\pi}\cdot\frac{1}{2}\left(\operatorname{ash}\varphi + \varphi\sqrt{1+\varphi^2}\right)\Bigg|_{\varphi_0}^{\varphi_1} = \frac{h}{2\pi}\cdot\frac{1}{2}\left(\varphi\sqrt{1+\varphi^2} + \ln\left(\varphi + \sqrt{1+\varphi^2}\right)\right)\Bigg|_{\varphi_0}^{\varphi_1} =$$

$$\left[\operatorname{ash}\varphi = \ln\left(\varphi + \sqrt{1+\varphi^2}\right)\right]$$

$$= \frac{h}{2\pi}\left(\frac{\varphi_1}{2}\sqrt{\varphi_1^2+1} + \frac{1}{2}\ln\left(\varphi_1 + \sqrt{\varphi_1^2+1}\right) - \frac{\varphi_0}{2}\sqrt{\varphi_0^2+1} - \frac{1}{2}\ln\left(\varphi_0 + \sqrt{\varphi_0^2+1}\right)\right) = L(\varphi_0, \varphi_1)$$